DECOMPOSITION OF THE ADJOINT REPRESENTATION OF THE SMALL QUANTUM sl_2.

VICTOR OSTRIK

1. Introduction.

1.1. Given a finite type root datum and a primitive root of unity $q = \sqrt[l]{1}$, G. Lusztig has defined in [Lu] a remarkable finite dimensional Hopf algebra $\mathfrak{u}$ over the cyclotomic field $\mathbb{Q}(\sqrt[l]{1})$. It is called a *restricted quantum universal enveloping algebra*, or a *small quantum group*.

Recall that for a Hopf algebra A with a coproduct Δ, antipode S, and counit ε the adjoint representation is defined in the following way. A is an A-bimodule with respect to left and right multiplication. Using the antipode we can consider A as $A \otimes A$-module. Combining this with the coproduct we get a new structure of A-module on A. It is called the *adjoint representation* and denoted by **ad**.

1.2. In this note we study the adjoint representation of $\mathfrak{u}$ in the simplest case of the root datum sl_2.

The semisimple part of this representation is of big importance in the study of local systems of conformal blocks in WZW model for $\hat{sl}_2$ at level $l-2$ in arbitrary genus. The problem of distinguishing the semisimple part is closely related to the problem of integral representation of conformal blocks (see [BFS]).

We find all the indecomposable direct summands of **ad** with multiplicities. To formulate the answer let us recall a few notations from the representation theory of $\mathfrak{u}$. The representation **ad** is naturally $\mathbb{Z}$-graded, so we consider the category $\mathcal{C}$ of $\mathbb{Z}$-graded $\mathfrak{u}$-modules. The simple modules in this category are parametrized by their highest weights which can assume arbitrary integer values. The simple module with a highest weight $\lambda \in \mathbb{Z}$ is denoted by $L(\lambda)$, and its indecomposable projective cover is denoted by $P(\lambda)$.

1.3. **Main theorem.** *Let $l \geq 3$ be an odd integer. The adjoint representation* **ad** *is isomorphic to the direct sum of the modules* $P(0), P(2), \dots, P(l-3); L(-l+1), L(-l+3), \dots, L(2l-4), L(2l-2)$ *with the following multiplicities:*

Date: December 1995.

(i) the multiplicity of $P(l-1) \simeq L(l-1)$ *is* l;
Let $i \in [0, \dots, \frac{l-3}{2}]$.
(ii) the multiplicity of $P(2i)$ *is* $\frac{l+1}{2} + i$;
(iii) the multiplicity of $L(2i)$ *is* $l-1-2i$;
(iiii) the multiplicities of $L(2l-2-2i)$ *and* $L(-2-2i)$ *are* $\frac{l-1}{2} - i$.

1.4. G. Lusztig has defined (see e.g. [Lu]) a *quantum enveloping algebra with divided powers* U containing $\mathfrak{u}$ as a Hopf subalgebra. Its finite dimensional irreducible representations are parametrized by their highest weights which can assume arbitrary nonnegative integer values. The simple module with a highest weight $\lambda \in \mathbb{N}$ is denoted by $\hat{L}(\lambda)$, and its indecomposable projective cover is denoted by $\hat{P}(\lambda)$. There is a natural restriction functor from the category of finite dimensional U-modules to $\mathcal{C}$. It appears that **ad** lies in its essential image.

Corollary. *The structure of* $\mathbb{Z}$*-graded* $\mathfrak{u}$*-module on* **ad** *can be lifted to the structure of* U*-module isomorphic to the direct sum of the modules* $\hat{P}(0), \hat{P}(2), \dots, \hat{P}(l-3); \hat{L}(0), \hat{L}(2), \dots, \hat{L}(2l-4), \hat{L}(2l-2)$ *with the following multiplicities:*
(i) the multiplicity of $\hat{P}(l-1) \simeq \hat{L}(l-1)$ *is* l;
Let $i \in [0, \dots, \frac{l-3}{2}]$.
(ii) the multiplicity of $\hat{P}(2i)$ *is* $\frac{l+1}{2} + i$;
(iii) the multiplicity of $\hat{L}(2i)$ *is* $l-1-2i$;
(iiii) the multiplicity of $\hat{L}(2l-2-2i)$ *is* $\frac{l-1}{2} - i$.

1.5. The author is very grateful to Michael Finkelberg; it is hard to overestimate his help with writing down the present work. Also I am happy to express my sincere gratitude to Thomas Kerler who taught me everything I know about **ad**.

2. Notations

2.1. In this section we recall the necessary facts about $\mathfrak{u}$ and its representations, following mainly [Lu].

Let l be an odd integer, $l > 1$. Let $q \in \mathbb{C}$ be a primitive $l-$th root of unity. Let $(i)_q = \frac{q^i - q^{-i}}{q - q^{-1}}$.

Let $\mathfrak{u}$ be an associative algebra over $\mathbb{C}$ with generators E, F, K, K^{-1} and relations:

$$KK^{-1} = K^{-1}K = 1;$$

$$KEK^{-1} = q^2 E, KFK^{-1} = q^{-2} F;$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}};$$
$$E^l = F^l = 0, K^l = 1.$$

The algebra $\mathfrak{u}$ is finite-dimensional and $\dim \mathfrak{u} = l^3$.

Let ω be an automorphism of the algebra $\mathfrak{u}$ given on generators by the formulas:

$$\omega(E) = F,\ \omega(F) = E,\ \omega(K) = K^{-1}.$$

The algebra $\mathfrak{u}$ is Hopf algebra with respect to coproduct Δ, antipode S and counit ε given by the formulas:

$$\Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \Delta(K) = K \otimes K;$$
$$S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1};$$
$$\varepsilon(E) = \varepsilon(F) = 0, \varepsilon(K) = 1.$$

2.2. Let $\mathcal{C}$ be a category of finite-dimensional $\mathbb{Z}$-graded $\mathfrak{u}$-modules $V = \bigoplus_{i \in \mathbb{Z}} V^i$ such that the following conditions hold:

(a) E is operator of degree 2, i.e. E acts from V^i to V^{i+2};
(b) F is operator of degree -2;
(c) K acts on V^i by multiplication by q^i.

The morphisms in category $\mathcal{C}$ are morphisms of $\mathfrak{u}$-modules compatible with $\mathbb{Z}$-grading.

2.3. We introduce the duality D on category $\mathcal{C}$. If $V \in \mathcal{C}$, then $D(V)$ is V^* as a vector space. The action of $x \in \mathfrak{u}$ on $D(V)$ is given by the formula $(xf)(v) = f(\omega S(x)v)$, where $f \in D(V), v \in V, S$ is the antipode.

2.4. Let us define the adjoint representation $\mathbf{ad} \in \mathcal{C}$ (see e.g. [LM]). Let x be an element of $\mathfrak{u}$. The adjoint action of generators is given by the folloing formulas:

$$ad(E)x = Ex - KxK^{-1}E = K[K^{-1}E, x],$$
$$ad(F)x = FxK - xFK = [F, x]K,$$
$$ad(K)x = KxK^{-1}.$$

2.5. Now we introduce $\mathbb{Z}$-grading on adjoint representation. We put $\deg(E) = 2, \deg(F) = -2, \deg(K) = 0$ and $\deg(ab) = \deg(a) + \deg(b)$ for any $a, b \in \mathfrak{u}$ such that $\deg(a), \deg(b)$ are defined. Note that all the weights of $\mathbf{ad}$ are even integers in the interval $[2 - 2l, \dots, 2l - 2]$.

2.6. It is known (see e.g. [LM] or [BFS]) that $D(\mathbf{ad}) \simeq \mathbf{ad}$.

3. $\mathfrak{u}$-MODULES.

3.1. It is easy to check that an element

$$X = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \tag{1}$$

lies in the center of algebra $\mathfrak{u}$ (see e.g. [Ke]). The element X is called Casimir element. It satisfies the following equation of degree l (see loc.cit.):

$$P(X) := \prod_{j\in\mathbb{Z}/l\mathbb{Z}} (X - b_j) = 0, \tag{2}$$

where $b_j = \frac{q^{j+1}+q^{-j-1}}{(q-q^{-1})^2}$ ($q^l = 1$, so q^j is well defined). In particular $b_j = b_{j'}$ if $j + j' = l - 2$. The root $b_{-1} = \frac{2}{(q-q^{-1})^2}$ of P has multiplicity 1, and the rest roots b_j have multiplicity 2.

3.2. Let $j \in \mathbb{Z}/l\mathbb{Z}$. Let $\mathcal{C}_j$ be a full subcategory of $\mathcal{C}$ such that $(X - b_j)$ acts nilpotently on objects of $\mathcal{C}_j$. In what follows we will identify $\mathcal{C}_j$ and $\mathcal{C}_{j'}$ if $j + j' = l - 2$.

3.3. Let us fix H'— a maximal subset of $\mathbb{Z}/l\mathbb{Z}$ with the following property: if $\{j, j'\}$ is two-element subset of H' then $j + j' \neq l - 2$. We have $H' = \{-1\} \cup H$, where $H = H' - \{-1\}$.

3.4. For any $j \in H$ we define the integers J, J' by the following properties:

(1) $0 \leq J < J' < l$;
(2) $J + J' = l - 2$;
(3) $(J - j)(J' - j) \equiv 0 \pmod{l}$.

3.5. The category $\mathcal{C}$ is a direct sum of subcategories $\mathcal{C}_j$ where j runs through H'.

3.6. We denote by $\mathfrak{u}^{\pm} \subset \mathfrak{u}$ the subalgebra generated by K, E (resp. K, F). For $\lambda \in \mathbb{Z}$ denote by $\mathbb{C}^{\pm}_{\lambda}$ the one-dimensional $\mathbb{Z}-$graded $\mathfrak{u}^{\pm}-$module of weight λ such that K acts as q^{λ} and E (resp. F) acts as zero on it. We denote by $M^{\mp}(\lambda)$ the $\mathbb{Z}-$graded $\mathfrak{u}-$module $\mathfrak{u} \otimes_{\mathfrak{u}^{\pm}} \mathbb{C}^{\pm}_{\lambda}$. The modules $M^{\pm}(\lambda)$ are called Verma modules. Let $M^{\pm}(\lambda) \ni v^{\pm}(\lambda) := 1 \otimes 1$. Let $V \in \mathcal{C}, v \in V^{\lambda}$ and $E \cdot v = 0$ (resp. $Fv = 0$). Then v is called an upper singular (resp. a lower singular) vector in V, and there exists a unique morphism $\phi : M^{\pm}(\lambda) \to V$ such that $\phi(v^{\pm}(\lambda)) = v$.

3.7. For each $\lambda \in \mathbb{Z}$ there is a unique up to isomorphism simple module $L(\lambda) \in \mathcal{C}$ with highest weight λ. The modules $L(\lambda_1)$ and $L(\lambda_2)$ are isomorphic iff $\lambda_1 \equiv \lambda_2 (\mathrm{mod}\ l)$. Indecomposable projective cover of $L(\lambda)$ will be denoted by $P(\lambda)$. We have $D(L(\lambda)) \simeq L(\lambda)$ and $D(P(\lambda)) \simeq P(\lambda)$. In particular $P(\lambda)$ is injective; $\dim \mathrm{Hom}(L(\lambda), P(\lambda)) = 1$.

3.8. The set of isomorphism classes of simple objects in category $\mathcal{C}_{-1}$ is $\{L(\lambda), \lambda \equiv -1 (\mathrm{mod}\ l)\}$. As $\mathfrak{u}$-modules without grading all the $L(\lambda)$ are isomorphic to one and the same $\mathfrak{u}$-module St (Steinberg module). It has dimension l. The category $\mathcal{C}_{-1}$ is semisimple. In particular $P(\lambda) = L(\lambda)$.

3.9. Let $j \in H$. The set of isomorphism classes of simple modules in $\mathcal{C}_j$ is $\{L(\lambda), \lambda \equiv J (\mathrm{mod}\ l) \text{ or } \lambda \equiv J' (\mathrm{mod}\ l)\}$. The modules $L(\lambda), \lambda \equiv J (\mathrm{mod}\ l)$ (resp. $\lambda \equiv J' (\mathrm{mod}\ l)$) are isomorphic as $\mathfrak{u}$-modules. Their dimension is $J+1$ (resp. $J'+1$).

The projective module $P(\lambda)$ admits a filtration $P(\lambda) \supset W(\lambda) \supset L(\lambda) \supset 0$ such that $P(\lambda)/W(\lambda) \simeq L(\lambda), W(\lambda)/L(\lambda) \simeq L(\lambda') \oplus L(\lambda'')$ where $\lambda' \neq \lambda''$, $\lambda' \equiv \lambda'' \equiv -2-\lambda\ (\mathrm{mod}\ l), |\lambda - \lambda'| < 2l > |\lambda - \lambda''|$. In particular $\dim P(\lambda) = 2l$.

3.10. It is easy to see from 2.5, 3.8 and 3.9 that all the simple subquotients of adjoint representation have the type $L(\lambda)$ where λ is an even integer from the interval $[1-l, \dots, 2l-2]$. Hence a projective module $P(\lambda)$ can be a subquotient of **ad** only if $\lambda \in 2\mathbb{Z} \cap [0, \dots, l-1]$. In particular each subcategory $\mathcal{C}_j$ contains only one isomorphism class of such projectives.

3.11. **Lemma.** *Suppose $V \in \mathcal{C}$ is indecomposable and the action of Casimir element X on V is not semisimple. Then there exists $\lambda \not\equiv -1\ (\mathrm{mod}\ l)$ such that $V \simeq P(\lambda)$.*

Proof. Casimir acts nonsemisimply on regular representation (see (2)). It follows that action on projective modules $P(\lambda), \lambda \not\equiv -1\ (\mathrm{mod}\ l)$ is not semisimple. It is easy to see that the space of eigenvectors of Casimir in $P(\lambda)$ is $W(\lambda)$.

Let $W \subset V$ be a maximal submodule of V such that X acts on W semisimply. Choose $0 \neq \varphi \in \mathrm{Hom}(V, L)$ where $L = L(\lambda)$ for some $\lambda \in \mathbb{Z}$ such that Ker φ contains W. We have a morphism $\psi \in \mathrm{Hom}(P, V)$ where $P = P(\lambda)$ such that the diagram is commutative:

$$\begin{tikzcd} P \arrow[d, "\psi"'] \arrow[dr] & & \\ V \arrow[r, "\varphi"] & L \arrow[r] & 0 \end{tikzcd}$$

If Ker $\psi \neq 0$ then X acts on Im ψ semisimply. Therefore W is not maximal. We have a contradiction. If Ker $\psi = 0$ then we have injection $P \hookrightarrow V$. From 3.7 follows that P is direct summand of V. The proof is complete. $\square$

4. The blocks of adjoint representation

In what follows we always will identify S_j and $S_{j'}$ where $j + j' = l - 2$ and S is an object, map, etc. Also we will identify S_J and S_j if $J \in \mathbb{Z}, J \equiv j \pmod{l}$.

4.1. The regular action of Casimir X (by multiplication) is an endomorphism of adjoint representation. This gives a decomposition of adjoint representation into *blocks* $\mathbf{ad} = \bigoplus\limits_{j \in H'} \mathbf{ad}_j$ where $(X - b_j)$ acts nilpotently on $\mathbf{ad}_j$. Let pr_j denote a projection onto $\mathbf{ad}_j$.

4.2. Let $j \in H$. Let $M_j = \mathrm{Ker}(X - b_j)$, $N_j = \mathbf{ad}_j \cap \mathrm{Im}(X - b_j)$. Each $\mathbf{ad}_j$ admits a filtration $\mathbf{ad}_j \supset M_j \supset N_j \supset 0$. The rest of this section is a computation of assotiated graded of this filtration. Evidently $\mathbf{ad}_j / M_j \simeq N_j$. It remains to compute $N_j, M_j/N_j$. It is convenient to put $N_{-1} = \mathbf{ad}_{-1}$.

4.3. Recall (see 2.2) that $\mathbf{ad}^0 \subset \mathbf{ad}$ denotes the zero weight space. Let $\mathbf{ad}^0_j = \mathbf{ad}_j \cap \mathbf{ad}^0$ (for all $j \in H'$), $N^0_j = N_j \cap \mathbf{ad}^0$, $M^0_j = M_j \cap \mathbf{ad}^0$ (for $j \in H$) and $N^0_{-1} = \mathbf{ad}^0_{-1}$. We will compute the action of $\mathrm{ad}(X)$ on $\mathbf{ad}^0_j$. We start with a computation of action of $\mathrm{ad}(X)$ on N^0_j.

4.4. **Lemma.** *We have*

(a) $\dim \mathbf{ad}_j = 2l^2$ *if* $j \in H$ *and* $\dim \mathbf{ad}_{-1} = l^2$;

(b) if $j \in H$ *then* $\dim N_j = \dim \mathbf{ad}_j / M_j = (J+1)^2 + (J'+1)^2$ *and* $\dim M_j / N_j = 4(J+1)(J'+1)$.

(c) $\dim \mathbf{ad}^{2m} = l(l - |m|)$ *for all* $m \in \mathbb{Z}$ *such that* $|m| < l$;

(d) $\dim \mathbf{ad}^0_j = 2l$ *for* $j \in H$ *and* $\dim N^0_j = l$ *for all* $j \in H'$.

(e) $\dim \mathbf{ad}^{2m}_j \geq 2(l - |m|)$ *if* $j \in H$ *and* $|m| \geq l - 1 - J$.

Proof. (a), (b), (c), (d) are trivial. Let us prove (e). Suppose $m > 0$. It is easy to see from consideration of $\mathfrak{u}$-action on Verma modules $M^+(J)$ and $M^+(J')$ that E^m acts nontrivially at least on $2(l - m)$ weights. By standard arguments with Vandermond determinant we obtain that the desired dimension is at least $2(l-m)$. The proof for $m < 0$ is similar. $\square$

4.5. The subspace $\mathbf{ad}^0 \subset \mathfrak{u}$ is a subalgebra of $\mathfrak{u}$. It is generated as algebra by K and X (see [Ke]). Moreover $\mathbf{ad}^0$ is a free module over a subalgebra generated by K (see loc. cit.). In particular we have $M_j^0 = N_j^0$.

We have

$$\mathrm{ad}(X)K^i = \frac{q^{2i-1} + q^{1-2i}}{(q-q^{-1})^2}K^i - (q^i - q^{-i})^2 XK^{i+1} + (i)_q(i+1)_q K^{i+2} \tag{3}$$

4.6. For $j \in H$ the elements $(X - b_j)pr_j K^i, i = 1, \dots, l$ (resp. $pr_{-1}K^i, i = 1, \dots, l$ for $j = -1$) form a basis of N_j^0. In this basis $\mathrm{ad}(X)$ acts as a lower-triangular matrix:

$$A(j) = \begin{pmatrix} b_0 & 0 & 0 & \dots & 0 \\ (q-q^{-1})^2 b_j & b_2 & 0 & \dots & 0 \\ (1)_q(2)_q & (q^2-q^{-2})^2 b_j & b_4 & \dots & 0 \\ 0 & (2)_q(3)_q & (q^3-q^{-3})^2 b_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_0 \end{pmatrix} \tag{4}$$

4.6.1. *Remark.* The vectors $pr_j K^i, i = 1, \dots, l; (X - b_j)pr_j K^i, i = l+1, \dots, 2l$ form a basis of $\mathbf{ad}_j^0$ $(j \in H)$.

4.7. Let $k \in 2\mathbb{Z} \cap [0, \dots, l-1]$. The eigenvalues of this matrix are b_k (with multiplicity 2 if $k \neq l-1$ and multiplicity 1 if $k = l-1$). Let $k \neq l-1$. It is obvious that there exists 1 or 2 eigenvectors of $A(j)$ corresponding to eigenvalue b_k. We have 2 eigenvectors iff the determinant $d(j,k)$ of matrix (see Lemma 6.1)

$$D(j,k) = \begin{pmatrix} (q^{k/2+1} - q^{-k/2-1})^2 b_j & b_{k+2} - b_k & 0 & \dots \\ (k/2+1)_q(k/2+2)_q & (q^{k/2+2} - q^{-k/2-2})^2 b_j & b_{k+4} - b_k & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots \end{pmatrix} \tag{5}$$

is equal to zero. It is easy to see that this determinant is a polynomial in b_j^2 of degree $\frac{l-1-k}{2}$. Since $b_j^2 = b_{j'}^2 \Rightarrow b_j = b_{j'}$, the polynomial $d(j,k)$ vanishes for at most $\frac{l-1-k}{2}$ values of $j \in H'$.

4.8. In this section we prove the following proposition:

Proposition. *For any $j \in H$ we have the following decomposition:*

$$N_j \simeq \bigoplus_{i=0}^{J}(L(2i) \oplus L(2i)) \oplus \bigoplus_{i=J+1}^{\frac{l-3}{2}} P(2i) \oplus L(l-1).$$

Proof. The proof proceeds by induction: we start from $j = \frac{l-3}{2}$, then proceed to $j = \frac{l-5}{2}$ etc.

It follows from 4.6 that for any $j \in H'$ the module N_j contains as subquotients $L(0), L(2), \ldots, L(l-3)$ with multiplicities 2 and $L(l-1)$ with multiplicity 1 (since only these modules have nontrivial zero weight space).

Lemma. *Let* $j+1 = \frac{l-1}{2}$*. Then* $N_j \simeq L(0) \oplus L(0) \oplus \ldots \oplus L(l-3) \oplus L(l-1)$.

Proof. Let us compute the dimensions. We have $\dim N_j = (\frac{l-1}{2})^2 + (\frac{l+1}{2})^2 = \frac{l^2+1}{2}$ (see Lemma 4.4(b)). On the other hand by the above we have $\dim N_j \geq 2\dim L(0) + 2\dim L(2) + \ldots + 2\dim L(l-3) + \dim L(l-1) = 2\cdot 1 + 2\cdot 3 + \ldots + 2\cdot(l-2) + l = \frac{l^2+1}{2}$. It follows that in this case N_j is a direct sum of $L(0), L(2), \ldots, L(l-3)$ with multiplicities 2 and $L(l-1)$ with multiplicity 1 (since $\mathrm{Ext}^1(L(\lambda), L(\mu)) = 0 \quad \forall \lambda, \mu \in \{0, 2, \ldots, l-1\}$). □

The Lemma implies that all eigenvalues of $A(\frac{l-3}{2})$ are semisimple. It follows from 4.6 that eigenvalue b_{l-3} is not semisimple for all the rest j. Therefore for $j \neq \frac{l-3}{2}$ the corresponding N_j contains projective submodule $P(l-3)$ (see 3.11).

Now let $j+1 = \frac{l-3}{2}$. Then $\dim N_j = \frac{l^2+9}{2}$. On the other hand $\dim N_j \geq 2\cdot 1 + \ldots + 2\cdot(l-4) + 2l + l = \frac{l^2+9}{2}$. So in this case N_j is a direct sum of $L(0), L(2), \ldots, L(l-5)$ with multiplicities 2, $L(l-1)$ with multiplicity 1, and $P(l-3)$. As above it follows that all the rest N_j contains projective submodules $P(l-5)$ and $P(l-3)$ etc. The Proposition is proved. □

4.8.1. **Corollary.** *We have:*

$$\mathbf{ad}_{-1} = N_{-1} = \bigoplus_{i=0}^{\frac{l-1}{2}} P(2i).$$

Proof. It follows from the proof of the Proposition 4.8 that all the eigenvalues of the matrix $A(-1)$, except for b_{l-1}, are not semisimple. Hence the result follows from the Lemma 3.11 and computation of dimensions. □

4.9. The Corollary 4.8.1 gives a decomposition of $\mathbf{ad}_{-1}$. So in what follows we will assume that $j \neq -1$ i.e. $j \in H$.

Lemma. *The module* M_j/N_j *is a direct sum of modules* $L(\lambda)$ *with multiplicity 2 where* λ *is even and satisfies one of the following conditions:*

(i) λ *lies in the interval* $[2(l-J+1), \ldots, 2l-2]$;

(ii) λ *lies in the interval* $[-2J-2, \ldots, -2]$.

Proof. Recall that $M_j^0 = N_j^0$. Hence M_j/N_j contains only subquotients $L(\lambda)$ where either $\lambda > l$ or $\lambda < 0$. By the Lemma 4.4(e), Proposition 4.8, and Corollary 4.8.1, we have $\dim \mathbf{ad}_j^{2m} \geq 2(l-|m|)$ if $j \in H$ and $\dim \mathbf{ad}_{-1}^{2m} \geq l-|m|$ for any $m \geq \frac{l+1}{2}$. It follows from the Lemma 4.4(c) that $\dim \mathbf{ad}_j^{2m} = 2(l-|m|)$ for any $j \in H$ and $m \geq \frac{l+1}{2}$.

Hence for any $m \geq \frac{l+1}{2}$ we have exactly two upper (resp. lower) singular vectors of weight $2m$ (resp. $-2m$). It follows that $\mathbf{ad}_j$ has two simple subquotients with highest weight $2m$ and two simple subquotients with lowest weight $-2m$ for any $m \geq \frac{l+1}{2}$. Thus $\mathbf{ad}_j$ has the following subquotients: $L(2i)$ where $i \in [0, \ldots, \frac{l-3}{2}]$ with multiplicities 4; $L(l-1)$ with multiplicity 2; $L(-2-2i)$ and $L(2l-2-2i)$, where $i \in [0, \ldots, \frac{l-3}{2}]$, with multiplicities 2. The computation of dimensions shows that these modules are all the subquotients of $\mathbf{ad}_j$. It follows from Proposition 4.8 that $[N_j : L(\lambda)] = 1$ if $\lambda = -2-2i$ and $\lambda = 2l-2-2i$, where $i \in [J+1, \ldots, \frac{l-3}{2}]$. Since $\mathbf{ad}_j/M_j \simeq N_j$, any simple subquotient of M_j/N_j is of the type $L(\lambda)$ where $\lambda \in [2(l-J+1), \ldots, 2l-2] \cup [-2J-2, \ldots, -2]$ is even. But for any λ, μ satisfying such conditions we have $\mathrm{Ext}^1(L(\lambda), L(\mu)) = 0$. The result follows. $\square$

4.9.1. *Remark.* It follows from the proof of the Lemma 4.9 that $\dim \mathbf{ad}_j^{2m} = 2(l-|m|)$ for any $j \in H$ and $m \in \mathbb{Z}, |m| < l$.

4.10. Let k be an even integer and $k \in [0, l-1]$. Let $\mathbf{ad}_j(k)$ be a summand corresponding to the subcategory $\mathcal{C}_k$ in $\mathbf{ad}_j$. Let $M_j(k) = M_j \cap \mathbf{ad}_j(k)$ and $N_j(k) = N_j \cap \mathbf{ad}_j(k)$. Let us summarize the results of the present section.

4.10.1. $\mathbf{ad}_{-1}(k) = P(k)$.

4.10.2. If $j \in H$ then

(a) $\mathbf{ad}_j(l-1) = L(l-1) \oplus L(l-1)$;

(b) if $k \geq 2J+2$ then $\mathbf{ad}_j(k) = P(k) \oplus P(k)$;

(c) if $k \leq 2J$ then $\mathbf{ad}_j(k)$ admits a fitration $\mathbf{ad}_j(k) \supset M_j(k) \supset N_j(k) \supset 0$ with the following associated graded factors:

$$N_j(k) \simeq L(k) \oplus L(k);$$

$$M_j(k)/N_j(k) \simeq L(2l-2-k) \oplus L(2l-2-k) \oplus L(-2-k) \oplus L(-2-k);$$

$$\mathbf{ad}_j(k)/M_j(k) \simeq L(k) \oplus L(k).$$

5. The proof of the main theorem

5.1. Let us find the multiplicities of projective submodules in $\mathbf{ad}_j$. Recall (see Remark 4.6.1) that the vectors $pr_jK^i, i = 1, \dots, l; (X - b_j)pr_jK^i, i = l+1, \dots, 2l$ form a basis of $\mathbf{ad}_j^0$. In this basis $\mathrm{ad}(X)$ acts as a block matrix

$$A'(j) = \begin{pmatrix} A(j) & 0 \\ B & A(j) \end{pmatrix}$$

where $A(j)$ is a matrix (4) and B is a matrix (see (3))

$$\begin{pmatrix} 0 & 0 & 0 & \dots \\ -(q-q^{-1})^2 & 0 & 0 & \dots \\ 0 & -(q^2-q^{-2})^2 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

5.2. Let b_k be a nonsemisimple eigenvalue of $A(j)$. Then a summand corresponding to the subcategory $\mathcal{C}_k$ in $\mathbf{ad}_j$ is a sum of 2 copies of projective $P(k)$.

5.3. Let b_k $(k \neq -1)$ be a semisimple eigenvalue of $A(j)$, i.e. $k \leq 2J$.

5.3.1. **Lemma.** *In this case* $\mathbf{ad}_j$ *contains projective module from category* $\mathcal{C}_k$.

Proof. It is enough to prove that the matrix $A'(j)$ has exactly 3 eigenvectors with eigenvalue b_k or, equivalently, that the matrix $A'(j) - b_k$ has corank 3. Let us denote by $\tilde{A}'(j,k)$ the matrix $A'(j,k) - b_k$ with the $i-$th and $(i+l)-$th columns divided by positive numbers $-(q^i - q^{-i})^2$ for any $i \in [1, \dots, l-1]$. In order to apply the Lemma 6.2 let us put $A' = \tilde{A}'(j,k)$. Then corresponding matrix D in notations of the Lemma 6.2 is the matrix $D(j,k)$ with columns divided by some positive numbers. In order to check the semisimplicity of the matrix D put $q = \exp(\pi i \frac{l+1}{l})$. Then conditions of Lemma 6.3 hold. Indeed, off-diagonal entries of $D(j,k)$ are either $(t)_q(t+1)_q$ or $b_{k+2t} - b_k = (t)_q(t+k+1)_q$. In both cases this entry is $(t_1)_q(t_2)_q$ where $t_1, t_2 \in [1, \dots, l-1]$ and one of t_1, t_2 is even and another is odd. But if $q = \exp(\pi i \frac{l+1}{l})$ and $t \in [1, \dots, l-1]$ then $(t)_q > 0 \Leftrightarrow t$ is odd. Finally note that the entries of D are the entries of $D(j,k)$ divided by some positive numbers. The Lemma is proved.$\square$

5.3.2. The above Lemma implies that $\mathbf{ad}_j(k)$ is a sum of a projective module $P(k)$ and some module $Y(k,j)$. It follows from 4.10 that $Y(k,j)$ admits a filtration of length 3 with the following associated graded factors: $L(k); L(-2-k) \oplus L(2l-2-k); L(k)$.

5.3.3. For any $0 \leq s < l$ an element $pr_j(K^{-1}E)^s$ is an upper singular vector in $\mathbf{ad}_j$. Let us prove that if $s \leq \frac{l-1}{2}$ then $(X-b_j)pr_j(K^{-1}E)^s \neq 0$. Consider the action of this element on $P(J')$. Then $(X-b_j)$ is a surjection onto $L(J') \subset P(j')$ and the desired result follows from the fact that $\dim L(J') \geq \frac{l+1}{2}$. Similarly the vector pr_jF^s is a lower singular vector in $\mathbf{ad}_j$, and $pr_jF^s \notin M_j$ if $s \leq \frac{l-1}{2}$. For $s = \frac{k}{2}$ we obtain that $\mathbf{ad}_j$ contains a submodule $L(k)$ which does not lie in M_j. Indeed, the submodules generated by $pr_j(K^{-1}E)^s$ and pr_jF^s coincide since $\mathbf{ad}_j(k)/M_j(k) \simeq L(k) \oplus L(k)$ and $\mathbf{ad}_j(k) \supset P(k) \not\subset M_j(k)$.

5.3.4. It follows that $Y(k,j) = Z(k,j) \oplus L(k)$ where $Z(k,j)$ contains a submodule $L(k)$. Hence $\mathbf{ad}(k) = \bigoplus_{j \in H'} \mathbf{ad}_j(k)$ is a direct sum of a few copies of $P(k)$ and $Y(k) = \bigoplus_{j \in H'} Y(k,j)$ where all the subquotients $L(k)$ of $Y(k)$ are the submodules of $Y(k)$. Now from the autoduality (see 2.6) we see that all the subquotients $L(k)$ are direct summands of $Y(k)$. Thus $Y(k)$ is a direct sum of its simple subquotients. Hence in the case 4.10.2 (c) (i.e. if $k \leq 2J$) we have that

$$\mathbf{ad}_j(k) = P(k) \oplus L(k) \oplus L(k) \oplus L(2l-2-k) \oplus L(-2-k)$$

This completes the proof of the Main Theorem. $\square$

6. Three matrix lemmas

The results of this section were used in the previous sections.

6.1. Let A be a $r \times r$ lower-triangular matrix:

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ \beta_1 & \alpha_2 & 0 & \dots & 0 \\ \gamma_1 & \beta_2 & \alpha_3 & \dots & 0 \\ 0 & \gamma_2 & \beta_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{r-1} & \alpha_r \end{pmatrix} \tag{6}$$

Lemma. *Let $\alpha_i = \alpha_j = \alpha$ for some $i < j$ and $\alpha_k \neq \alpha$ for $k \neq i, j$. The matrix A has 2 different eigenvectors with eigenvalue α iff the determinant of $(j-i) \times (j-i)$ matrix*

$$D = \begin{pmatrix} \beta_i & \alpha_{i+1}-\alpha & 0 & \dots & 0 \\ \gamma_i & \beta_{i+1} & \alpha_{i+2}-\alpha & \dots & 0 \\ 0 & \gamma_{i+1} & \beta_{i+2} & \dots & 0 \\ 0 & 0 & \gamma_{i+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{j-2} & \beta_{j-1} \end{pmatrix} \qquad (7)$$

vanishes.

Proof. Clear.□

6.2. Suppose $\alpha_i = \alpha_j$ iff $i + j = r + 1$. Let A' be the following matrix

$$A' = \begin{pmatrix} A & 0 \\ B' & A \end{pmatrix}$$

where

$$B' = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Lemma. *Suppose the matrix A above has 2 eigenvectors with eigenvalue α_i. Suppose the matrix D is semisimple of corank 1. Then the matrix $A' - \alpha_i$ has corank 3.*

Proof. It suffices to consider the case $i = 1, j = r, \alpha_1 = \alpha_r = 0$. Deleting two rows and columns consisting of zeros we obtain a matrix

$$D' = \begin{pmatrix} D & 0 \\ E & D \end{pmatrix}$$

where E is a unit matrix. We have to prove that corank of D' is equal to 1. Let $\mathrm{Im}(D)$ (resp. $\mathrm{Im}(D')$) denote the linear space generated by columns of D (resp. D'). Let $pr : \mathrm{Im}(D') \to \mathbb{C}^{r-1}$ be a map forgetting the last $r - 1$ coordinates. Then $pr(\mathrm{Im}(D')) = \mathrm{Im}(D)$ has dimension $r - 2$. Let us prove that $\mathrm{Ker}(pr)$ has dimension $r - 1$. Indeed, $\mathrm{Ker}(pr) \supset \mathrm{Im}(D)$ and $\mathrm{Ker}(pr)$ contains the kernel of operator D. Since D is semisimple $\mathrm{Im}(D) \not\supset \mathrm{Ker}(D)$. The result follows. □

6.3. **Lemma.** *Suppose D is a real matrix, and all the off-diagonal entries are negative. Then D is semisimple.*

Proof. It is easy to see that conjugating matrix D by some diagonal matrix we can obtain a symmetric matrix. □

References

[BFS] R.Bezrukavnikov, M.Finkelberg, V.Schechtman, *Localization of* $\mathfrak{u}$*-modules. V. The modular structure on the category* $\mathcal{FS}$, Preprint, to appear.

[Ke] T.Kerler, *Mapping class group actions on quantum doubles,* Comm. Math. Phys. **168** (1995), 353-388.

[Lu] G.Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebras,* J. of AMS **3** (1990), 257-296.

[LM] V.Lyubashenko, S.Majid, *Braided groups and quantum Fourier transform*, Preprint DAMTP/91-26.

Independent University of Moscow, PO Box 230, Moscow, 117463, Russia